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A CHARACTERIZATION OF NONSTANDARD LIFTINGS OF MEASURABLE FUNCTIONS AND STOCHASTIC PROCESSES

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ABSTRACT
The concept of a lifting of a measurable function in nonstandard stochastic analysis is studied. Several characterizations of liftings are given. The theory of liftings is related to standard measure-theoretical results about the Lebesgue space $L^1([0,1])$. A method to construct liftings is presented.

0. Introduction

Liftings can be thought of as nonstandard versions of measurable functions or stochastic processes. The concept of a lifting of a measurable function is touched in Loeb's paper [6] where he shows how to convert internal $*$-additive measures into σ-additive external ones. Anderson [1], Keisler [4], and Lindström [5] used liftings to develop the nonstandard theory of stochastic integration.

Liftings have properties nice to work with (e.g. hyperfiniteness in [4], [5]) and resemble their standard versions in a large part of their properties. Using liftings, integrals of functions can be written as hyperfinite sums (see [4], [6], [1]), stochastic differential equations as hyperfinite stochastic difference equations (cf. [4]).

Keisler [4] gave a nonstandard characterization of Lebesgue measurability of functions $[0,1]\to\mathbf{R}$ by showing that measurable functions are exactly the ones having a lifting.

In this paper we develop a number of conditions characterizing the property of being a lifting. The characterization problem for liftings is typical for nonstandard analysis in the sense that it is one instance of the general problem of taking "standard parts": by working with nonstandard objects to study properties of standard objects it often becomes necessary to show that an internally constructed nonstandard object (whose external properties are not fully known) represents a standard entity.

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In the first section we characterize liftings by a variation principle. The problem of showing that something is a lifting is reduced to a simple calculation.

The approximative nature of liftings is made precise in the second section where we use the results of the first section to characterize liftings as nearstandard elements with respect to some suitable space of measurable functions related to $L^1([0,1])$. This new aspect of liftings opens the way for nonstandard proofs of standard results about the set of measurable functions on $[0,1]$ using known facts about liftings; we show how fundamental properties of $L^1([0,1])$ such as completeness and separability are reflected in the nonstandard theory. It turns out that there is a close relationship between theorems about liftings and standard measure-theoretical results about the set of measurable functions on $[0,1]$; one can also use such results to give new proofs of known facts about liftings.

As a consequence of the nearstandardness characterization of liftings we show that with respect to integration, liftings reflect the properties of their standard versions. Moreover, liftings are even characterized by that property which shows that they are the right nonstandard representatives of standard functions in a hyperfinite integration theory.

In the third section we give an explicit formula to construct liftings of measurable functions $[0,1]\to\mathbf{R}$ which we also apply to stochastic processes.

We assume that the reader is familiar with the basics of both nonstandard analysis and standard measure theory. For background see [7] or [2], respectively.

We essentially adopt the notation of [4]. Throughout this paper we assume that T is a "hyperfinite time interval" of the form $T=\{K/H \mid K\in {}^*\mathbf{N}_0, K<H\}$ where $H\in {}^*\mathbf{N}-\mathbf{N}$ ($\mathbf{N}_0$ denotes the set $\mathbf{N}\cup\{0\}$), Ω is a hyperfinite set, and (M,d) a complete separable metric space. The standard part of elements $t\in {}^*\mathbf{R}$, $p\in {}^*M$, etc. is denoted by ${}^\circ t$, ${}^\circ p$, etc. For any hyperfinite set Γ, the internal cardinality of Γ is denoted by $|\Gamma|$. For $r\in {}^*\mathbf{R}$, $[r]$ denotes the largest hyperinteger less than or equal to r. For any hyperfinite set Γ, $\bar{P}_\Gamma$ denotes the normalized internal counting measure on Γ and P_Γ the (completed) associated Loeb measure, i.e. P_Γ is the completion of the unique σ-additive measure $\tilde{P}_\Gamma$ on the σ-algebra generated by the internal subsets of Γ such that $\tilde{P}_\Gamma(B)={}^\circ\bar{P}_\Gamma(B)$ for all internal $B\subseteq\Gamma$ (see [6]). When referring to a measure on a hyperfinite set Γ, we always mean P_Γ. On $[0,1]$ and $[0,1]^2$ we work with the Lebesgue measures which are denoted by λ and λ^2, respectively. On $\Omega\times[0,1]$ we consider the completion of $P_\Omega\times\lambda$.

We now state some results from the literature on Loeb measures.

DEFINITION A (cf. [1], [4], [6]). Let $(\mathcal{M}, d)$ be a separable metric space. A function $V:\Omega\to{}^*\mathcal{M}$ is said to be a lifting of $v:\Omega\to\mathcal{M}$ if V is internal and ${}^\circ V(\omega)=v(\omega)$ a.s. (P_Ω).

$F:T\to{}^*\mathcal{M}$ is a lifting of $f:[0,1]\to\mathcal{M}$ if F is internal and ${}^\circ F(t)=f({}^\circ t)$ a.s. (P_T).

$X:\Omega\times T\to{}^*\mathcal{M}$ is a lifting of $x:\Omega\times[0,1]\to\mathcal{M}$ if X is internal and ${}^\circ X(\omega,t)=x(\omega,{}^\circ t)$ a.s. $(P_{\Omega\times T})$.

REMARK B. In the second section it is sometimes more natural to work with liftings that are defined on the whole of ${}^*[0,1]$ $(\Omega\times{}^*[0,1])$ instead of T $(\Omega\times T)$. Such a concept of a lifting can be defined correspondingly (cf. [1]); the Loeb measure associated to ${}^*\lambda$ on ${}^*[0,1]$ then replaces P_T. All results quoted below and proved in the paper hold correspondingly for this modified concept. However, the hyperfinite notion of a lifting as defined above turns out to be sufficient for the nonstandard treatment of measurable functions $[0,1]\to M$ discussed in §2.

THEOREM C (cf. [1], [4], [6]). *Let $(\mathcal{M}, d)$ be a separable metric space.*

(i) $v:\Omega\to\mathcal{M}$ *has a lifting if and only if* v *is measurable.*

(ii) $f:[0,1]\to\mathcal{M}$ *has a lifting if and only if* f *is measurable.*

(iii) $x:\Omega\times[0,1]\to\mathcal{M}$ *has a lifting if and only if* x *is measurable.*

THEOREM D (cf. [6]). *Let* Γ *be a hyperfinite set. For any* P_Γ*-measurable set* $V\subseteq\Gamma$,

$$P_\Gamma(V)=\sup\{{}^\circ\bar{P}_\Gamma(U)\mid U\subseteq V,\ U\ \text{internal}\}.$$

THEOREM E (cf. [1]). *The standard part mapping* $\mathrm{st}:T\to[0,1]$, $t\to{}^\circ t$ *is measurable and measure-preserving, i.e. for any Lebesgue measurable* $B\subseteq[0,1]$, $\mathrm{st}^{-1}(B)$ *is Loeb measurable and*

$$P_T(\mathrm{st}^{-1}(B))=\lambda(B).$$

THEOREM F (Fubini theorem for Loeb measure, cf. [3]). *Let* Ω_1 *and* Ω_2 *be hyperfinite and let* $x:\Omega_1\times\Omega_2\to\mathbf{R}$ *be bounded and* $P_{\Omega_1\times\Omega_2}$*-measurable. Then:*

(i) *For almost all* $\omega_1\in\Omega_1$, $x(\omega_1,\cdot)$ *is Loeb measurable on* Ω_2.

(ii) *The function* $y(\omega_1)=\int x(\omega_1,\omega_2)dP_{\Omega_2}(\omega_2)$ *is Loeb measurable on* Ω_1.

(iii) $\int x(\omega_1,\omega_2)dP_{\Omega_1\times\Omega_2}(\omega_1,\omega_2)=\int(\int x(\omega_1,\omega_2)dP_{\Omega_2}(\omega_2))dP_{\Omega_1}(\omega_1)$.

We use the following extensions of the concepts of integrability and S-integrability (cf. [1]) to functions with values in arbitrary metric spaces $\mathcal{M}$ $({}^*\mathcal{M}$, respectively).

For any probability space Π and metric space $(\mathcal{M}, d)$, a function $f:\Pi\to\mathcal{M}$ is said to be *integrable with respect to* d if f is measurable and for any $p \in \mathcal{M}$ the function $d(f(\pi), p): \Pi \to \mathbf{R}$ is integrable. This is the case for any $p \in \mathcal{M}$ if and only if it is the case for some $p \in \mathcal{M}$. The concept of *internal integrability with respect to* ${}^*d$ is defined accordingly for internal ${}^*\mathcal{M}$-valued functions on internal probability spaces.

Now let (A, μ) be an internal probability space. We will mostly work with hyperfinite spaces (with normalized internal counting measure) so that internal integrals over A become hyperfinite sums. Let $(\mathcal{M}, d)$ be a metric space. A function $J: A \to {}^*\mathcal{M}$ which is internally integrable with respect to ${}^*d$ is said to be *S-integrable with respect to* d if for any (i.e. for some) $p \in \mathcal{M}$ the function ${}^*d(J(\omega), {}^*p): A \to {}^*\mathbf{R}$ is S-integrable, i.e. for any μ-measurable $B \subseteq A$,

$$\mu(B) \approx 0 \Rightarrow \int_B {}^*d(J(\omega), {}^*p) d\mu \approx 0.$$

For $M = \mathbf{R}$, d the standard metric of $\mathbf{R}$, this means that J is S-integrable: let $J: A \to {}^*\mathbf{R}$ be internally integrable. Obviously, J is S-integrable with respect to the standard metric of $\mathbf{R}$ iff for any μ-measurable $B \subseteq A$

$$\mu(B) \approx 0 \Rightarrow \int_B J(\omega) d\mu \approx 0,$$

which says that J is S-integrable (see [1], also Theorem H).

The following two theorems about functions which are S-integrable with respect to d are straightforward generalizations of corresponding results for S-integrable functions:

THEOREM G (cf. [1], [6]). *Let $(\mathcal{M}, d)$ be a separable, metric space.*

(i) *$v: \Omega \to \mathcal{M}$ has a lifting which is S-integrable with respect to d if and only if v is integrable with respect to d.*

(ii) *$f: [0,1] \to \mathcal{M}$ has a lifting which is S-integrable with respect to d if and only if f is integrable with respect to d.*

(iii) *$x: \Omega \times [0,1] \to \mathcal{M}$ has a lifting which is S-integrable with respect to d if and only if x is integrable with respect to d.*

THEOREM H [cf. [1]). *Let (A, μ) be an internal probability space and $(\mathcal{M}, d)$ a metric space. Let $J: A \to {}^*\mathcal{M}$ be internally integrable with respect to ${}^*d$. J is S-integrable with respect to d if and only if for any $p \in \mathcal{M}$, ${}^\circ\!\int d(J(\omega), {}^*p) d\mu$ exists and*

$${}^\circ\!\int {}^*d(J(\omega), {}^*p) d\mu = \lim_{n \to \infty} {}^\circ\!\int {}^*d(J(\omega), {}^*p) \wedge n \, d\mu$$

if and only if this holds for some $p \in \mathcal{M}$.

THEOREM I (cf. [1]). *Let $f:[0,1]\to\mathbf{R}$ be integrable and let $F:T\to{}^*\mathbf{R}$ be an S-integrable lifting of f. Then*

$$^\circ\sum_{t\in T}F(t)|T|^{-1}=\int_0^1 f(t)dt.$$

In the following considerations T can easily be replaced by any other hyperfinite S-dense (i.e. the image under st is $[0,1]$) subset of $^*[0,1]$. $[0,1]$ can be replaced by $[0,1]^k$ for $k\in\mathbf{N}$.

The treatment of internal functions $F:T\to{}^*M$ and hyperfinite stochastic processes $X:\Omega\times T\to{}^*M$ in the first section is parallel. Most results can be formulated in corresponding ways for both types of functions so that we will present the proofs only for internal $X:\Omega\times T\to{}^*M$ and add "$(F:T\to{}^*M)$" to indicate that internal functions $F:T\to{}^*M$ are treated correspondingly. The case for internal $F:T\to{}^*M$ can also be directly reduced to that of internal $X:\Omega\times T\to{}^*M$ by defining an ω-independent process $X(\omega,t):=F(t)$ $(\omega\in\Omega, t\in T)$.

1. Characterization of liftings by a variation principle

We use the following variation principle to characterize the smoothness of liftings:

DEFINITION 1. For any $\tau\in{}^*[0,1]$ let

$$\Delta_\tau:=\{(t,t')\in T^2 \mid |t-t'|\leqq\tau\}.$$

Assume $X:\Omega\times T\to{}^*M$ is internal. X is said to *satisfy the lifting condition* if for every positive infinitesimal τ

$$P_{\Omega\times\Delta_\tau}(X(\omega,t)\approx X(\omega,t'))=1.^\dagger$$

Correspondingly, an internal function $F:T\to{}^*M$ *satisfies the lifting condition* if for every positive infinitesimal τ

† We use the term

$$P_{\Omega\times\Delta_\tau}(X(\omega,t)\approx X(\omega,t'))$$

as abbreviation for

$$P_{\Omega\times\Delta_\tau}(\{(\omega,(t,t'))\in\Omega\times\Delta_\tau \mid X(\omega,t)\approx X(\omega,t')\}).$$

More generally, if (Π,ν) is a probability space and S a statement about the elements ω of a set $\Pi'\supseteq\Pi$, we write $\nu(S(\omega))$ for $\nu(\{\omega\in\Pi \mid S(\omega)\})$. This convention is correspondingly applied to internal probability spaces.

$$P_{\Delta_\tau}(F(t) \approx F(t')) = 1.$$

DEFINITION 2. An internal function $X : \Omega \times T \to {}^*M$ is called *smooth* if there exists a Loeb measurable $U \subseteq \Omega \times T$ such that $P_{\Omega\times T}(U) = 1$ and for all $\omega \in \Omega$ and $t, t' \in T$

$$\text{if } (\omega, t) \in U \text{ and } (\omega, t') \in U \text{ and } t \approx t' \text{ then } X(\omega, t) \approx X(\omega, t').$$

An internal function $F : T \to {}^*M$ is called *smooth* if there exists a Loeb measurable $U \subseteq T$ such that $P_T(U) = 1$ and for all $t, t' \in U$

$$\text{if } t \approx t' \text{ then } F(t) \approx F(t').$$

Obviously, $X : \Omega \times T \to {}^*M$ $(F : T \to {}^*M)$ is a lifting of some function $x : \Omega \times [0,1] \to M$ $(f : [0,1] \to M)$ if and only if X (F) is smooth and has values a.s. nearstandard in ${}^*M$.

Given an internal function $F : T \to {}^*M$, we shall henceforth simply say "F is a lifting" to mean that F is a lifting of some function $f : [0,1] \to M$. Correspondingly, we say that an internal function $X : \Omega \times T \to {}^*M$ "is a lifting" if X is a lifting of some function $x : \Omega \times [0,1] \to M$.

THEOREM 3. *Let $X : \Omega \times T \to {}^*M$ $(F : T \to {}^*M)$ be internal. X (F) is a lifting if and only if it satisfies the lifting condition and has values a.s. nearstandard in ${}^*M$.*

PROOF. We show in several steps that X is smooth exactly in case it satisfies the lifting condition. The first step is

PROPOSITION 4. *Let $X : \Omega \times T \to {}^*M$ $(F : T \to {}^*M)$ be internal. X (F) satisfies the lifting condition if it is smooth.*

PROOF. Let X be smooth and $U \subseteq \Omega \times T$ such that $P_{\Omega\times T}(U) = 1$ and for all $\omega \in \Omega$ and $t, t' \in T$

$$\text{if } (\omega, t) \in U \text{ and } (\omega, t') \in U \text{ and } t \approx t' \text{ then } X(\omega, t) \approx X(\omega, t').$$

Let τ be a positive infinitesimal and let

$$W(\tau) := \{(\omega, (t, t')) \in \Omega \times \Delta_\tau \mid (\omega, t) \in U \text{ and } (\omega, t') \in U\}.$$

It is easy to show that $P_{\Omega\times T}(U) = 1$ implies $P_{\Omega\times\Delta_\tau}(W(\tau)) = 1$. Furthermore, for any $(\omega, (t, t')) \in W(\tau)$, $X(\omega, t) \approx X(\omega, t')$, so X satisfies the lifting condition.

DEFINITION 5. An internal function $X : \Omega \times T \to {}^*M$ is *approximable by step* *functions* if for any positive $\varepsilon \in \mathbf{R}$ there is an $n \in \mathbf{N}$ such that for all $m \in \mathbf{N}_0$, $m < n$ there exists an internal function $\chi_{n,m} : \Omega \to {}^*M$ satisfying

$$\bar{P}_{\Omega\times T}({}^*d(X(\omega,t),\chi_{n,[nt]}(\omega)) \geqq \varepsilon) \leqq \varepsilon.$$

An internal function $F : T \to {}^*M$ is *approximable by step functions* if for any positive $\varepsilon \in \mathbf{R}$ there is an $n \in \mathbf{N}$ such that for all $m \in \mathbf{N}_0$, $m < n$ there exists $p_{n,m} \in {}^*M$ satisfying

$$\bar{P}_T({}^*d(F(t), p_{n,[nt]}) \geqq \varepsilon) \leqq \varepsilon.$$

PROPOSITION 6. *Let $X : \Omega \times T \to {}^*M$ ($F : T \to {}^*M$) be internal. X (F) is approximable by step functions if X (F) satisfies the lifting condition.*

We first show:

LEMMA 7. *Let $X : \Omega \times T \to {}^*M$ be internal and let $n \in \mathbf{N}$, $\varepsilon \in \mathbf{R}$, $\varepsilon > 0$. Then for each $m \in \mathbf{N}_0$, $m < n$ there exists $t_{n,m} \in T \cap {}^*[m/n,(m+1)/n)$ such that*

$$\bar{P}_{\Omega\times T}({}^*d(X(\omega,t), X(\omega,t_{n,[nt]})) \geqq \varepsilon) \leqq 2\bar{P}_{\Omega\times\Delta_\tau}({}^*d(X(\omega,t),X(\omega,t')) \geqq \varepsilon)$$

(*correspondingly for $F : T \to {}^*M$*).

PROOF. Fix $\varepsilon \in \mathbf{R}$, $\varepsilon > 0$ and $n \in \mathbf{N}$. For any $m \in \mathbf{N}_0$, $m < n$ define

$$S_{n,m} := \{t \in T \mid m/n \leqq t < (m+1)/n\}, \qquad S_n := \bigcup_{m<n} S^2_{n,m}.$$

Then $S_n \subseteq \Delta_{1/n}$ and $|S_n| \geqq |\Delta_{1/n}| \cdot 2^{-1}$, hence

(1) $\bar{P}_{\Omega\times S_n}({}^*d(X(\omega,t),X(\omega,t')) \geqq \varepsilon) \leqq 2\bar{P}_{\Omega\times\Delta_{1/n}}({}^*d(X(\omega,t),X(\omega,t')) \geqq \varepsilon)$.

Since for any $m \in \mathbf{N}_0$, $m < n$

$$\bar{P}_{\Omega\times S^2_{n,m}}({}^*d(X(\omega,t),X(\omega,t')) \geqq \varepsilon)$$

$$= \sum_{t' \in S_{n,m}} \bar{P}_{\Omega\times S_{n,m}}({}^*d(X(\omega,t),X(\omega,t')) \geqq \varepsilon) \cdot |S_{n,m}|^{-1},$$

for any $m \in \mathbf{N}_0$, $m < n$ there must be $t_{n,m} \in S_{n,m}$ such that

(2) $\bar{P}_{\Omega\times S_{n,m}}({}^*d(X(\omega,t),X(\omega,t_{n,m})) \geqq \varepsilon) \leqq \bar{P}_{\Omega\times S^2_{n,m}}({}^*d(X(\omega,t),X(\omega,t')) \geqq \varepsilon)$.

Thus

$$\bar{P}_{\Omega\times T}({}^*d(X(\omega,t),X(\omega,t_{n,[nt]})) \geqq \varepsilon)$$

$$\approx n^{-1} \sum_{m<n} \bar{P}_{\Omega\times S_{n,m}}({}^*d(X(\omega,t),X(\omega,t_{n,m})) \geqq \varepsilon)$$

by (2) $$\leqq n^{-1}\sum_{m<n}\bar{P}_{\Omega\times S^2_{n,m}}({}^*d(X(\omega,t),X(\omega,t'))\geqq\varepsilon)$$

$$\approx \bar{P}_{\Omega\times S_n}({}^*d(X(\omega,t),X(\omega,t'))\geqq\varepsilon)$$

by (1) $$\leqq 2\bar{P}_{\Omega\times\Delta_{1/n}}({}^*d(X(\omega,t),X(\omega,t'))\geqq\varepsilon).$$

Proposition 6 follows from Lemma 7 since the lifting condition implies that for any positive $\varepsilon\in\mathbf{R}$

$$P_{\Omega\times\Delta_{1/n}}({}^*d(X(\omega,t),X(\omega,t'))\geqq\varepsilon)\to 0 \qquad \text{as } n\to\infty.$$

The last step in the proof of Theorem 3 is

PROPOSITION 8. *Let $X:\Omega\times T\to{}^*M$ $(F:T\to{}^*M)$ be internal. X (F) is smooth if it is approximable by step functions.*

PROOF. If X is approximable by step functions, then for any positive $\varepsilon\in\mathbf{R}$, there is an $n\in\mathbf{N}$ and $V\subseteq\Omega\times T$ such that $P_{\Omega\times T}(V)\geqq 1-\varepsilon$ and for all $\omega\in\Omega$, $t,t'\in T$,

(1) if $(\omega,t),(\omega,t')\in V$ and $[nt]=[nt']$, then ${}^*d(X(\omega,t),X(\omega,t'))\leqq 2\varepsilon$.

For any $n\in\mathbf{N}$ and $\delta\in(0,1]$, the set $T^{(n,\delta)}:=T\backslash(\bigcup_{m=0}^{n-1}{}^*[m/n,m/n+\delta))$ has the property that whenever $t,t'\in T^{(n,\delta)}$, $|t-t'|\leqq\delta$, then $[nt]=[nt']$. Now, for any positive ε, $\delta\in\mathbf{R}$, if n and $V\subseteq\Omega\times T$ are chosen according to (1), the set $V':=V\cap\Omega\times T^{(n,\delta)}$ has measure not less than $1-\varepsilon-n\delta$ and

$$\text{if } \omega\in\Omega,\ t,t'\in T,\ (\omega,t),(\omega,t')\in V', \text{ and } |t-t'|\leqq\delta,$$
$$\text{then } {}^*d(X(\omega,t),X(\omega,t'))\leqq 2\varepsilon.$$

Rephrasing this, for any positive $\varepsilon\in\mathbf{R}$ there exist $U_\varepsilon\subseteq\Omega\times T$ and $\delta_\varepsilon>0$ such that $P_{\Omega\times T}(U_\varepsilon)\geqq 1-\varepsilon$ and for all $\omega\in\Omega$, $t,t'\in T$

$$\text{if } (\omega,t),(\omega,t')\in U_\varepsilon \text{ and } |t-t'|\leqq\delta_\varepsilon, \text{ then } {}^*d(X(\omega,t),X(\omega,t'))\leqq\varepsilon.$$

Now let

$$U=\liminf_{i\to\infty}U_{1/i^2}=\bigcup_{j=1}^{\infty}\bigcap_{i=j}^{\infty}U_{1/i^2}.$$

Elementary calculations show $P_{\Omega\times T}(U)=1$ and, whenever $\omega\in\Omega$, $t,t'\in T$ such that $(\omega,t),(\omega,t')\in U$ and $t\approx t'$, then ${}^*d(X(\omega,t),X(\omega,t'))\leqq 1/i^2$ for all $i>j$ for some $j\in\mathbf{N}$, i.e. $X(\omega,t)\approx X(\omega,t')$.

By Propositions 4, 6, and 8, the proof of Theorem 3 is now complete.

REMARK 9. Theorem 3 shows that the set of internal functions $F:T\to{}^*M$

which are liftings is Borel, and in fact Π_2^0, over the internal sets (similarly for internal functions $X:\Omega\times T\to{}^*M$): it is easily seen that an internal function $F:T\to{}^*M$ satisfies the lifting condition iff for every $m\in\mathbf{N}$ there exists $n\in\mathbf{N}$ such that

$$\bar{P}_{\Delta_\tau}({}^*d(F(t),F(t'))\leqq 1/m)\geqq 1-1/m \qquad (\tau\in{}^*[0,1/n]). \tag{1}$$

Furthermore, if $p_1,p_2,\cdots$ is a countable dense subset of M and $S_{1/m}(p_i)$ denotes the set $\{q\in M \mid d(p_i,q)\leqq 1/m\}$ $(m,i\in\mathbf{N})$, then F has values a.s. nearstandard in ${}^*M$ iff for every $m\in\mathbf{N}$ there exists $n\in\mathbf{N}$ such that

$$\bar{P}_T\left(F^{-1}\left(\bigcup_{i=1}^{n}{}^*S_{1/m}(p_i)\right)\right)\geqq 1-1/m. \tag{2}$$

Thus the set of liftings is given by $(\bigcap_{m=1}^{\infty}\bigcup_{n=1}^{\infty}A_{n,m})\cap(\bigcap_{m=1}^{\infty}\bigcup_{n=1}^{\infty}B_{n,m})$ where $A_{n,m}$ for $n,m\in\mathbf{N}$ is the (internal) set of internal functions $F:T\to{}^*M$ satisfying (1) and $B_{n,m}$ is the (internal) set of internal functions $F:T\to{}^*M$ satisfying (2).

We now turn to the treatment of functions which are S-integrable with respect to d.

For internal $X:\Omega\times T\to{}^*M$ define the "smoothness function" $G_X:{}^*[0,1]\to{}^*\mathbf{R}$ by

$$G_X(\tau):=\sum_{\omega\in\Omega}\sum_{(t,t')\in\Delta_\tau}{}^*d(X(\omega,t),X(\omega,t'))\cdot|\Delta_\tau|^{-1}\cdot|\Omega|^{-1}.$$

Similarly, for internal $F:T\to{}^*M$,

$$G_F(\tau):=\sum_{(t,t')\in\Delta_\tau}{}^*d(F(t),F(t'))\cdot|\Delta_\tau|^{-1}.$$

G_X and G_F are internal.

Definition 10. An internal function $X:\Omega\times T\to{}^*M$ *satisfies the lifting condition in integral form* if

$$G_X(\tau)\approx 0 \qquad \text{whenever } \tau\in{}^*[0,1],\quad \tau\approx 0.$$

An internal function $F:T\to{}^*M$ *satisfies the lifting condition in integral form* if

$$G_F(\tau)\approx 0 \qquad \text{whenever } \tau\in{}^*[0,1],\quad \tau\approx 0.$$

An easy consequence of Tchebycheff's inequality is

Proposition 11. *The lifting condition in integral form implies the lifting condition.*

Conversely, we show:

PROPOSITION 12. *Assume $X:\Omega\times T\to {}^*M$ $(F:T\to {}^*M)$ is S-integrable with respect to d and satisfies the lifting condition. Then X (F) satisfies the lifting condition in integral form.*

PROOF. Straightforward arguments show that S-integrability with respect to d of X implies that the function $\Omega\times\Delta_\tau\to {}^*\mathbf{R}$:

$$(\omega,(t,t'))\to {}^*d(X(\omega,t),X(\omega,t'))$$

is S-integrable on $\Omega\times\Delta_\tau$ for any $\tau\in {}^*[0,1]$.

Fix $\tau\approx 0$ and let

$$V:=\{(\omega,(t,t'))\in\Omega\times\Delta_\tau \mid X(\omega,t)\approx X(\omega,t')\}.$$

By assumption, $P_{\Omega\times\Delta_\tau}(V)=1$. For any internal $W\subseteq V$

$$0\approx G_X(\tau)\restriction_W:=\sum_{(\omega,(t,t'))\in W} {}^*d(X(\omega,t),X(\omega,t'))|\Delta_\tau|^{-1}\cdot|\Omega|^{-1}.$$

Using Theorem D, a typical saturation argument shows that there is an internal $V'\subseteq\Omega\times\Delta_\tau$ such that $P_{\Omega\times\Delta_\tau}(V\,\triangle\,V')=0$ and

$$(1)\qquad 0\approx G_X(\tau)\restriction_{V'}:=\sum_{(\omega,(t,t'))\in V'} {}^*d(X(\omega,t),X(\omega,t'))|\Delta_\tau|^{-1}\cdot|\Omega|^{-1}.$$

Furthermore, since $P_{\Omega\times\Delta_\tau}(\Omega\times\Delta_\tau-V')=P_{\Omega\times\Delta_\tau}(\Omega\times\Delta_\tau-V)=0$ and ${}^*d(X(\omega,t),X(\omega,t'))$ is S-integrable on $\Omega\times\Delta_\tau$,

$$(2)\qquad 0\approx G_X(\tau)\restriction_{\Omega\times\Delta_\tau-V'}:=\sum_{(\omega,(t,t'))\in\Omega\times\Delta_\tau-V'} {}^*d(X(\omega,t),X(\omega,t'))|\Delta_\tau|^{-1}\cdot|\Omega|^{-1}.$$

(1) and (2) prove $G_X(\tau)\approx 0$.

As a consequence of Theorem 3 and Propositions 11 and 12 we get:

COROLLARY 13. *Let $X:\Omega\times T\to {}^*M$ $(F:T\to {}^*M)$ be internal.*

(i) X (F) is a lifting if it satisfies the lifting condition in integral form and has values a.s. nearstandard in ${}^*M$.

(ii) X (F) satisfies the lifting condition in integral form and has values a.s. nearstandard in ${}^*M$ if it is a lifting which is S-integrable with respect to d.

EXAMPLE 14 (Brownian motion, cf. [1]). Let $\Omega=\{-1,1\}^T$, $X(\omega,t)=\sum_{0<s\le t}\omega(s)|T|^{-1/2}$ $(\omega\in\Omega, t\in T)$. With respect to $\mathbf{R}$ with the standard metric we then have

$$\text{(i)} \qquad G_X(\tau) \leqq \sum_{(t,t')\in\Delta_\tau} \left(\sum_{\omega\in\Omega} (X(\omega,t)-X(\omega,t'))^2 |\Omega|^{-1} \right)^{1/2} |\Delta_\tau|^{-1}$$

(by Schwartz' inequality)

$$= \sum_{(t,t')\in\Delta_\tau} |t-t'|^{1/2} |\Delta_\tau|^{-1} \approx 0$$

for any $\tau \approx 0$, i.e. X satisfies the lifting condition in integral form

$$\text{(ii)} \qquad \sum_{\omega\in\Omega}\sum_{t\in T} X^2(\omega,t)|T|\cdot|\Omega|^{-1} = \sum_{t\in T} t\cdot|T|^{-1} \approx \frac{1}{2} < \infty,$$

so X has values a.s. finite (i.e. nearstandard in $^*\mathbf{R}$) by Tchebycheff's inequality.

Hence, X is a lifting of some (measurable) function by Corollary 13.

2. Liftings as nearstandard objects

We now introduce spaces of measurable and with respect to d integrable functions. Let $\mathbf{M}_M([0,1])$ denote the set of measurable functions $[0,1]\to M$. We define $D_{\mathbf{M}}:(\mathbf{M}_M([0,1]))^2\to\mathbf{R}$ by

$$D_{\mathbf{M}}(f,g):=\inf\{\eta\in\mathbf{R} \mid \lambda(d(f(t),g(t))\geqq\eta)\leqq\eta\}.$$

Elementary considerations show that $D_{\mathbf{M}}$ is well-defined and has the properties of a pseudo-metric. $D_{\mathbf{M}}(f,g)=0$ is equivalent to $f(t)=g(t)$ a.s. We turn $\mathbf{M}_M([0,1])$ into a metric space $(\bar{\mathbf{M}}_M([0,1]),\bar{D}_{\mathbf{M}})$ by setting

$$(\bar{\mathbf{M}}_M([0,1]),\bar{D}_{\mathbf{M}}):=(\mathbf{M}_M([0,1]),D_{\mathbf{M}})/\sim$$

where $\sim$ denotes the equivalence relation on $\mathbf{M}_M([0,1])$ defined by

$$f\sim g \qquad \text{iff } D_{\mathbf{M}}(f,g)=0.$$

For any $f\in\mathbf{M}_M([0,1])$ we let $\bar{f}$ denote the equivalence class of f in $\bar{\mathbf{M}}_M([0,1])$ with respect to $\sim$. For functions $F\in{}^*\mathbf{M}_M$ the equivalence class of F in $^*\bar{\mathbf{M}}_M([0,1])$ is denoted by $\bar{F}$.

We say $f:[0,1]\to M$ is an n-step function if it is constant on each interval $[m/n,(m+1)/n)$ $(m\in\mathbf{N}_0, m<n)$. Correspondingly, for $H\in{}^*\mathbf{N}-\mathbf{N}$, $F:{}^*[0,1]\to{}^*M$ is an H-step function if it is constant on each interval $[K/H,(K+1)/H)$ $(K\in{}^*\mathbf{N}_0, K<H)$.

For any $F:T\to{}^*M$ there is an associated $|T|$-step function $\hat{F}:{}^*[0,1]\to{}^*M$ defined by $\hat{F}(t)=F(t^*)$ where t^* is the largest element of T which is $\leqq t$ $(t\in{}^*[0,1])$.

The next theorem shows that an internal function $F : T \to {}^*M$ is a lifting if and only if the associated $|T|$-step function is nearstandard in ${}^*\mathbf{M}_M([0,1])$:

THEOREM 15. *An internal function* $F : T \to {}^*M$ *is a lifting of* $f : [0,1] \to M$ *iff* ${}^*D_{\mathbf{M}}(\hat{F}, {}^*f) \approx 0$ *(i.e.* $\bar{f}$ *is the standard part of* $\bar{\hat{F}}$ *with respect to* $\bar{D}_{\mathbf{M}}$*).*

(Notice that, if F is a lifting of f, then $\hat{F} \in {}^*\mathbf{M}_M([0,1])$ and $f \in \mathbf{M}_M([0,1])$ by Theorem C, so ${}^*D_{\mathbf{M}}(\hat{F}, {}^*f)$ is defined.)

PROOF. Suppose F is a lifting of f. By Propositions 4 and 6, F is approximable by step functions, i.e. for any positive $\varepsilon \in \mathbf{R}$, there is an $n \in \mathbf{N}$ such that for all $m \in \mathbf{N}_0$, $m < n$ there exists $p_{n,m} \in {}^*M$ so that

$$(*) \qquad \bar{P}_T({}^*d(F(t), p_{n,[nt]}) \geqq \varepsilon) \leqq \varepsilon.$$

It is easily shown that, since F has values a.s. nearstandard in ${}^*M$, the $p_{n,m}$ can be chosen standard (i.e. $\in M$).

Since F lifts f, this implies

$$\lambda(\{t \in [0,1] \mid d(f(t), p_{n,[nt]}) > \varepsilon\}) \leqq \varepsilon.$$

Application of transfer to this inequality and thinking of F in $(*)$ as a $|T|$-step function yields

$${}^*\lambda({}^*d(\hat{F}(t), f(t)) > 2\varepsilon) \leqq \varepsilon.$$

Since ε was arbitrary, this means that ${}^*D_{\mathbf{M}}(\hat{F}, {}^*f) \approx 0$.

For the converse assume that $F : T \to {}^*M$ is internal, $f \in \mathbf{M}_M([0,1])$, and $D_{\mathbf{M}}(\hat{F}, {}^*f) \approx 0$. By Theorem C, f has a lifting $G : T \to {}^*M$. The first part of this theorem shows ${}^*D_{\mathbf{M}}(\hat{G}, {}^*f) \approx 0$, hence by assumption ${}^*D_{\mathbf{M}}(\hat{G}, \hat{F}) \approx 0$. This means that for any positive $\varepsilon \in \mathbf{R}$

$${}^*\lambda({}^*d(\hat{G}(t), \hat{F}(t)) \geqq \varepsilon) \leqq \varepsilon.$$

Since both $\hat{G}$ and $\hat{F}$ are $|T|$-step functions this is equivalent to

$$\bar{P}_T({}^*d(G(t), F(t)) \geqq \varepsilon) \leqq \varepsilon,$$

i.e. $G(t) \approx F(t)$ a.s. (P_T), so F also is a lifting of f.

We now turn to the treatment of functions which are integrable with respect to d. Let $L_M([0,1])$ be the set of functions $[0,1] \to M$ which are integrable with respect to d. We define a pseudo-metric D_L on $L_M([0,1])$ as follows:

$$D_L(f, g) := \int_0^1 d(f(t), g(t))dt \qquad (f, g \in L_M([0,1])).$$

Obviously, for $f, g \in L_M([0,1])$, $D_L(f,g)=0$ iff $f \sim g$. For $f \in L_M([0,1])$ and $g \in \mathbf{M}_M([0,1])$, $f \sim g$ implies $g \in L_M([0,1])$, i.e. $L_M([0,1])$ is closed under the relation $\sim$. $(L_M([0,1]), D_L)$ is turned into a metric space by setting

$$(\bar{L}_M([0,1]), \bar{D}_L) = (L_M([0,1]), D_L)/\sim.$$

For $M = \mathbf{R}$ with the standard metric, $\bar{L}_M([0,1])$ is the Lebesgue space $L^1([0,1])$ of integrable functions on $[0,1]$ (with the metric induced by the standard norm on $L^1([0,1])$).

The following analogue of Theorem 15 shows that an internal function $T \to {}^*M$ is a lifting which is S-integrable with respect to d if and only if the associated $|T|$-step function is nearstandard in ${}^*L_M([0,1])$:

THEOREM 16. *Let $F: T \to {}^*M$ be internal and let $f:[0,1] \to M$. F is a lifting of f which is S-integrable with respect to d iff ${}^*D_L(\hat{F}, {}^*f) \approx 0$ (i.e. $\bar{f}$ is the standard part of $\bar{\hat{F}}$ with respect to $\bar{D}_L$).*

To prove Theorem 16 we use:

LEMMA 17. *For any $f \in L_M([0,1])$, ${}^*f$ is S-integrable with respect to d on $[0,1]$.*

PROOF. Using Theorem H and standard measure-theoretical arguments.

PROOF OF THEOREM 16. This is a consequence of Theorem 15.

Assume that F is a lifting of f which is S-integrable with respect to d. Theorem G implies $f \in L_M([0,1])$, so ${}^*f$ is S-integrable with respect to d by Lemma 17.

S-integrability with respect to d of F on T implies S-integrability of $\hat{F}$ on ${}^*[0,1]$, so the function ${}^*d(\hat{F}(t), {}^*f(t)) : {}^*[0,1] \to {}^*\mathbf{R}$ is S-integrable on ${}^*[0,1]$. Together with ${}^*D_{\mathrm{M}}(F, {}^*f) \approx 0$, which follows from Theorem 15, this implies ${}^*D_L(\hat{F}, {}^*f) \approx 0$: using the definition of D_{M} we see that ${}^*D_{\mathrm{M}}(\hat{F}, {}^*f) \approx 0$ implies that for some positive $\varepsilon \in {}^*\mathbf{R}$, $\varepsilon \approx 0$,

$${}^*\lambda(A) \leqq \varepsilon \qquad \text{for } A := \{t \in {}^*[0,1] \mid {}^*d(\hat{F}(t), {}^*f(t)) \geqq \varepsilon\}.$$

We now have

$$\int_{{}^*[0,1]-A} {}^*d(\hat{F}(t), {}^*f(t)) d^*\lambda \leqq \varepsilon \approx 0$$

by definition of A, and

$$\int_A {}^*d(F(t), {}^*f(t)) d^*\lambda \approx 0$$

by S-integrability of ${}^*d(\hat{F}(t), {}^*f(t))$ on ${}^*[0,1]$. Hence ${}^*D_L(\hat{F}, {}^*f) \approx 0$.

Conversely, assume ${}^*D_L(\hat{F}, {}^*f) \approx 0$. Using Tchebycheff's inequality, it follows that ${}^*D_M(\hat{F}, {}^*f) \approx 0$, so F is a lifting of f by Theorem 15. ${}^*D_L(\hat{F}, {}^*f) \approx 0$ and S-integrability with respect to d of ${}^*f$ which follows from Lemma 17 imply that also $\hat{F}$ and thus F are S-integrable with respect to d.

REMARK 18. Theorems 15 and 16 together with Theorem 3 and Corollary 13 give a criterion for nearstandardness in ${}^*\bar{\mathbf{M}}_M([0,1])$ and ${}^*\bar{L}_M([0,1])$ for $|T|$-step functions. By working with liftings defined on ${}^*[0,1]$ instead of T (see Remark B) this criterion can be formulated in general form applicable to any function in ${}^*\mathbf{M}_M([0,1])$ or ${}^*L_M([0,1])$, respectively.

Theorems 15, 16 and C, G hold for any hyperfinite time interval T of the form $T = \{K/H \mid K \in \mathbf{N}_0, K < H\}$, $H \in {}^*\mathbf{N} - \mathbf{N}$. Consequently, we get the following approximation results:

COROLLARY 19. *For any measurable $f : [0,1] \to M$ and positive $\varepsilon \in \mathbf{R}$, there is an $n \in \mathbf{N}$ and an n-step function $g : [0,1] \to M$ such that*

$$\lambda(d(f(t), g(t)) \geqq \varepsilon) \leqq \varepsilon.$$

COROLLARY 20. *For any $f \in L_M([0,1])$ and positive $\varepsilon \in \mathbf{R}$,* there is an $n \in \mathbf{N}$ *and an n-step function $g : [0,1] \to M$ such that*

$$\int_0^1 d(f(t), g(t))dt \leqq \varepsilon.$$

Using separability of M, Corollaries 19 and 20 show that the spaces $(\bar{\mathbf{M}}_M([0,1]), \bar{D}_M)$ and $(\bar{L}_M([0,1]), \bar{D}_L)$ are separable; a countable dense subset is in either case given by the set of step functions with values in a fixed countable dense subset of M.

As another example for a standard result we prove the completeness of the space $(\bar{L}_M([0,1]), \bar{D}_L)$ by nonstandard methods. The completeness of $(\bar{\mathbf{M}}_M([0,1]), \bar{D}_M)$ can be proved analogously.

PROPOSITION 21. *Let $(F_i)_{i \in \mathbf{N}}$ be a sequence of internal functions $T \to {}^*M$. Assume $F : T \to {}^*M$ is internal and*

$$^{\circ*}D_L(F_i, F) \to 0 \qquad \textit{as } i \to \infty.$$

Then

(a) *if all F_i satisfy the lifting condition in integral form, so does F;*

(b) *if all F_i have values a.s. nearstandard in ${}^*M$, so does F;*

(c) *if all* F_i *are* S*-integrable with respect to* d, *so is* F.

PROOF. (a) follows from the inequality

$$
\begin{aligned}
|G_{F_i}(\tau)-G_F(\tau)| &\leqq \sum_{(t,t')\in\Delta_\tau} |{}^*d(F_i(t),F_i(t'))-{}^*d(F(t),F(t'))|\,|\Delta_\tau|^{-1} \\
&\leqq \sum_{(t,t')\in\Delta_\tau} {}^*d(F_i(t),F(t))|\Delta_\tau|^{-1}+{}^*d(F_i(t'),F_i(t'))|\Delta_\tau|^{-1} \\
&\leqq 2\sum_T {}^*d(F_i(t),F(t))\cdot(2[\tau|T|]+1)\cdot|\Delta_\tau|^{-1} \\
&\leqq 4\sum_T {}^*d(F_i(t),F(t))|T|^{-1} \\
&= 4{}^*D_L(\hat{F}_i,\hat{F}) \qquad (\tau\in{}^*[0,\tfrac{1}{2}],\ i\in\mathbf{N}).
\end{aligned}
$$

(b) Standard arguments show that

$$
{}^{\circ *}D_L(F_i,F)\to 0 \qquad \text{as } i\to\infty
$$

implies that for almost all $t\in T$ for any positive $\varepsilon\in\mathbf{R}$ there exists $i\in\mathbf{N}$ such that ${}^*d(F_i(t),F(t))<\varepsilon$. If all F_i $(i\in\mathbf{N})$ have values a.s. nearstandard in ${}^*M$ this shows that for almost all $t\in T$ for any positive $\varepsilon\in\mathbf{R}$ there exists $p\in M$ such that ${}^*d(F(t),{}^*p)<\varepsilon$, hence for almost all $t\in T$, $F(t)$ is nearstandard by completeness of M.

(c) follows from the inequality

$$
\sum_{t\in B} {}^*d(F(t),{}^*p)|T|^{-1}\leqq \sum_{t\in B} {}^*d(F_i(t),{}^*p)|T|^{-1}+{}^*D_L(\hat{F}_i,\hat{F})
$$

for any internal $B\subseteq T$, $p\in M$, $i\in\mathbf{N}$.

THEOREM 22. $(\bar{L}_M([0,1]),\bar{D}_L)$ *is complete.*

PROOF. Let $(\bar{f}_i)_{i\in\mathbf{N}}$ be a Cauchy sequence in $\bar{L}_M([0,1])$, $(F_i)_{i\in\mathbf{N}}$ a sequence of respective liftings which are S-integrable with respect to d existing by Theorem G. By Theorem 16, for all $i\in\mathbf{N}$

$$
(*) \qquad {}^*D_L(\hat{F}_i,{}^*f_i)\approx 0,
$$

so $(\bar{\hat{F}}_i)_{i\in\mathbf{N}}$ is still a Cauchy sequence in ${}^*\bar{L}_M([0,1])$ in the sense that for any positive $\varepsilon\in\mathbf{R}$ there is an $n_0\in\mathbf{N}$ such that

$$
{}^*D_L(\hat{F}_i,\hat{F}_j)<\varepsilon \qquad (i,j>n_0).
$$

By saturation, there is an internal $F: T \to {}^*M$ such that

$$(**) \qquad {}^{\circ *}D_L(\hat{F}_i, \hat{F}) \to 0 \qquad \text{as } i \to \infty.$$

By Proposition 21, the properties characterizing by Corollary 13 liftings which are S-integrable with respect to d are preserved under such limits, so, since all F_i are liftings which are S-integrable with respect to d, so is F. Let $f \in L_M([0,1])$ be a function lifted by F. Then

$${}^*D_L(\hat{F}, {}^*f) \approx 0$$

by Theorem 16. This, (*), and (**) show that $\bar{f}$ is a limit of $(\bar{f}_i)_{i \in \mathbf{N}}$ in $(\bar{L}_M([0,1]), \bar{D}_L)$.

REMARK 23. For this proof it is by no means essential that we work with hyperfinite liftings $T \to {}^*M$. In fact, the proof becomes slightly simpler if we have a more general notion of a lifting and Theorem 15 according to Remark 18. But, as shown by this proof, the concept of a hyperfinite lifting $T \to {}^*M$ is powerful enough since such liftings are "S-dense" in the set of nearstandard elements of ${}^*L_M([0,1])$ as shown by Theorem 16.

Intuitively, in the nonstandard theory liftings $T \to {}^*M$ replace approximating sequences of step functions and Theorem G (ii) plays a role similar to that of the combination of Corollary 20 and Theorem 22.

As a consequence of Theorem 15 we get a relationship between the three different types of liftings occurring in Definition A.

For any measurable $x: \Omega \times [0,1] \to M$ let $\tilde{x}: \Omega \to \bar{\mathbf{M}}_M([0,1])$ be defined by

$$\tilde{x}(\omega) := \overline{x(\omega, \cdot)},$$

i.e. $x(\omega)$ is the equivalence class with respect to D_M of the function $x(\omega, \cdot): [0,1] \to M$.

For any internal $X: \Omega \times T \to {}^*M$ define $\tilde{X}: \Omega \to {}^*\bar{\mathbf{M}}_M([0,1])$ by

$$\tilde{X}(\omega) := \overline{X\widehat{(\omega, \cdot)}}.$$

For any $x: \Omega \times [0,1] \to M$ which is integrable with respect to d, $\tilde{x}$ has values a.s. in $\bar{L}_M([0,1])$.

THEOREM 24. *Let $X: \Omega \times T \to {}^*M$ be internal and let $x: \Omega \times [0,1] \to M$ be measurable. X is a lifting of x iff $\tilde{X}$ is a lifting of $\tilde{x}$.*

PROOF. By definition of $\tilde{x}$ and $\tilde{X}$ and Theorem 15, we are done if we show:

an internal function $X:\Omega\times T\to {}^*M$ is a lifting of a measurable function $x:\Omega\times[0,1]\to M$ iff for almost all $\omega\in\Omega$, $X(\omega,\cdot)$ lifts $x(\omega,\cdot)$.

The "only-if-part" follows from Theorem F by definition of a lifting.

If for almost all $\omega\in\Omega$, $X(\omega,\cdot)$ is a lifting of $x(\omega,\cdot)$, then by Theorem 3 for almost all $\omega\in\Omega$ for any positive infinitesimal τ for almost all $(t,t')\in\Delta_\tau$, $X(\omega,t)\approx X(\omega,t')$ and for almost all $\omega\in\Omega$, $X(\omega,\cdot)$ has values a.s. nearstandard in ${}^*M$. So for any positive infinitesimal τ for almost all $(\omega,(t,t'))\in\Omega\times\Delta_\tau$, $X(\omega,t)\approx X(\omega,t')$ since by internality of X for any positive infinitesimal τ the set

$$\{(\omega,(t,t'))\in\Omega\times\Delta_\tau \mid X(\omega,t)\approx X(\omega,t')\}$$

is $P_{\Omega\times\Delta_\tau}$-measurable.

Similarly, it follows that X has values a.s. nearstandard in ${}^*M$ by showing that

$$\{(\omega,t)\in\Omega\times T \mid X(\omega,t)\text{ is nearstandard}\}$$

is $P_{\Omega\times T}$-measurable.

Hence, X is a lifting by Theorem 3. So, if for almost all $\omega\in\Omega$, $X(\omega,\cdot)$ is a lifting of $x(\omega,\cdot)$, then X is a lifting of some measurable function $y:\Omega\times[0,1]\to M$. Using the first part of this proof and measurability of x it follows that X also lifts x.

COROLLARY 25. *If $x:\Omega\times[0,1]\to M$ is measurable, then $\tilde{x}:\Omega\to\bar{\mathbf{M}}_M([0,1])$ is measurable.*

PROOF. By Theorem C (i), (iii), and Theorem 24.

REMARK 26. In this and the last section we parallelly treated measurable functions by stochastic methods and with respect to d integrable functions by integral methods. The second treatment (i.e. by integrals) can also be used to study the set of measurable functions $[0,1]\to M$ and liftings $T\to{}^*M$ by bringing into account that in the preceding results the metric d is variable. By considering a bounded metric d' on M equivalent to given d (for instance, $d'=d\wedge 1$) S-integrability assumptions become automatically true whereas the sets of measurable functions $[0,1]\to M$ and liftings $T\to{}^*M$ remain unchanged. So we have, for example, the following consequence of Corollary 13: an internal function $F:T\to{}^*M$ is a lifting (with respect to d) iff F has values a.s. nearstandard in ${}^*M$ and satisfies the lifting condition in integral form with respect to d', i.e.

$$\sum_{\Delta_\tau}{}^*d'(F(t),F(t'))\cdot|\Delta_\tau|^{-1}\approx 0\qquad(\tau\in{}^*[0,1],\tau\approx 0).$$

As another example we mention that a metric on the set of measurable functions $[0,1]\to M$ can be obtained by setting

$$D'_M(f,g)=\int_0^1 d'(f(t),g(t))dt \qquad (f,g\in L_M([0,1])).$$

We now follow up another aspect of Theorem 16. Theorem 16 says that with respect to "integration properties" liftings are infinitesimally close to their standard versions. More concretely, we show, improving Theorem I:

THEOREM 27. *Let $f:[0,1]\to M$ be integrable with respect to d and let $F:T\to {}^*M$ be internal. The following are equivalent:*

(i) *F is a lifting of f which is S-integrable with respect to d;*

(ii) *for any continuous $h:M\times[0,1]\to\mathbf{R}$ satisfying*

$$|h(p,t)-h(q,t)|\leqq d(p,q) \qquad \textit{for all } t\in[0,1],\quad p,q\in M,$$

$$^\circ\sum_{t\in T} {}^*h(F(t),t)|T|^{-1} \textit{ exists and equals } \int_0^1 h(f(t),t)dt.$$

Note that for any h with the above properties the function $h(f(t),t)$: $[0,1]\to\mathbf{R}$ is integrable.

PROOF. (i) $\Rightarrow$ (ii). Let h with the above properties be given. Define the functions $F':T\to{}^*\mathbf{R}$ and $f':[0,1]\to\mathbf{R}$ by

$$F'(t):={}^*h(F(t),t), \qquad f'(t):=h(f(t),t).$$

So

$$\begin{aligned}\left|\sum_{t\in T}{}^*h(F(t),t)|T|^{-1}-\int_0^1 h(f(t),t)dt\right| &= \left|\int_{^*[0,1]}(\hat{F}'(t)-{}^*f'(t))dt\right|\\ &\leqq \int_{^*[0,1]}|\hat{F}'(t)-{}^*f'(t)|\,dt\\ &\leqq \int_{^*[0,1]}{}^*d(\hat{F}(t),{}^*f(t))dt\\ &\qquad\text{by the properties of } h\\ &={}^*D_L(\hat{F},{}^*f)\\ &\approx 0 \quad\text{by Theorem 16.}\end{aligned}$$

(ii) $\Rightarrow$ (i). Let $\varepsilon\in\mathbf{R}$, $\varepsilon>0$. By Corollary 20, for some $n\in\mathbf{N}$ there is an n-step function $g:[0,1]\to M$ such that $D_L(g,f)<\varepsilon$. By (ii), for any continuous $z:[0,1]\to\mathbf{R}$ and $p\in M$

$$^\circ\sum_{t\in T} {}^*d(F(t), {}^*p)\cdot {}^*z(t)\cdot |T|^{-1} = \int_0^1 d(f(t), p)\cdot z(t)dt.$$

Hence for any $m \in \mathbf{N}_0$, $m < n$, by approximating $1_{[m/n,(m+1)/n]} : [0,1] \to [0,1]$ pointwise by continuous $z \geqq 1_{[m/n,(m+1)/n]}$, $z : [0,1] \to [0,1]$ we get

$$^\circ\sum_{t\in T\cap^*[m/n,(m+1)/n]} {}^*d(F(t), {}^*g(m/n))|T|^{-1} \leqq \int_{m/n}^{(m+1)/n} d(f(t), g(m/n))dt$$

and thus

$$^\circ\sum_{t\in T} {}^*d(F(t), {}^*g(t))|T|^{-1} = {}^\circ D_L(\hat{F}, \widehat{{}^*g\restriction_T}) < \varepsilon$$

where ${}^*g\restriction_T$ is the restriction of ${}^*g$ to T. Since g is an n-step function, ${}^\circ D_L(\widehat{{}^*\hat{g}\restriction_T}, {}^*g) = 0$. Consequently, ${}^\circ D_L(\hat{F}, {}^*f) < 2\varepsilon$. Since ε was arbitrary, ${}^\circ D_L(\hat{F}, {}^*f) = 0$, i.e. F is a lifting of f which is S-integrable with respect to d by Theorem 16.

Similarly, liftings of measurable functions $f : [0,1] \to M$ can be characterized by stochastic means. We note that we also have the following characterization (see Remark 26):

THEOREM 28. *Let $f : [0,1] \to M$ be measurable and $F : T \to {}^*M$ internal. The following are equivalent*:

(i) *F is a lifting of f*;

(ii) *for any bounded continuous $h : M \times [0,1] \to \mathbf{R}$*

$$^\circ\sum_{t\in T} h(F(t), t)|T|^{-1} = \int_0^1 h(f(t), t)dt.$$

PROOF. (i) $\Rightarrow$ (ii) follows from the corresponding part of Theorem 27 since ${}^*h(F(t), t) : T \to {}^*\mathbf{R}$ is an S-integrable lifting of $h(f(t), t) : [0,1] \to \mathbf{R}$ for any bounded continuous $h : M \times [0,1] \to \mathbf{R}$.

(ii) $\Rightarrow$ (i) follows similarly as the corresponding part of Theorem 27 by working with the bounded metric $d \wedge 1$ on M instead of d.

Theorem 27 has the following consequence for functions $f \in L_M([0,1])$ (Theorem I is strong enough for the following):

COROLLARY 29. *Any $f \in L_M([0,1])$ has the following property*:

$$(\lambda^2(\mathrm{st}\,\Delta_{1/n}))^{-1} \int_{\mathrm{st}\,\Delta_{1/n}} d(f(t), f(t'))d(t,t') \to 0 \qquad \text{as } n \to \infty.$$

PROOF. By "sectionwise" application of Theorem 27. Show that, given a lifting F of f which is S-integrable with respect to d, the sum function $T \to {}^*\mathbf{R}$:

$$t \to \sum_{\substack{t' \in T \\ |t'-t| \leq 1/n}} {}^*d(F(t), F(t'))|T|^{-1}$$

is an S-integrable lifting of the function $[0,1] \to \mathbf{R}$:

$$t \to \int_{\{t' \mid |t-t'| \leq 1/n\}} d(f(t), f(t'))dt'$$

for any $n \in \mathbf{N}$, using Theorem 27. This proves that for any $n \in \mathbf{N}$

$$(\lambda^2(\operatorname{st} \Delta_{1/n}))^{-1} \int_{\Delta_{1/n}} d(f(t), f(t'))d(t, t')$$

$$= {}^\circ\sum_{(t,t') \in \Delta_{1/n}} {}^*d(F(t), F(t')) \cdot |\Delta_{1/n}|^{-1} = {}^\circ G_F(1/n).$$

The claim then follows from the fact that ${}^\circ G_F(1/n) \to 0$ as $n \to \infty$ which is a consequence of Corollary 13(ii).

3. Construction of liftings

We now introduce a method to construct liftings of measurable functions $f : [0,1] \to \mathbf{R}$.

PROPOSITION 30. *For any $F \in {}^*L_{\mathbf{R}}([0,1])$ let $L_F : T \to {}^*\mathbf{R}$ be defined by*

$$L_F(t) := |T| \cdot \int_t^{t+1/|T|} F(s)ds.$$

If ${}^*D_L(F, {}^*f) \approx 0$ for $f \in L_{\mathbf{R}}([0,1])$, then L_F is an S-integrable lifting of f.

PROOF. We show ${}^*D_L(\hat{L}_F, {}^*f) \approx 0$, so the claim follows from Theorem 16. By Theorems G and 16, there is an internal function $G : T \to {}^*\mathbf{R}$ satisfying ${}^*D_L(\hat{G}, {}^*f) \approx 0$. Hence

$${}^*D_L(\hat{G}, \hat{L}_F) \leq \sum_{t \in T} |G(t) - L_F(t)| \, |T|^{-1}$$

$$= \sum_{t \in T} \left| |T| \int_t^{t+1/|T|} \hat{G}(s)ds - |T| \int_t^{t+1/|T|} F(s)ds \right| |T|^{-1}$$

$$\leq \sum_{t \in T} \int_t^{t+1/|T|} |\hat{G}(s) - F(s)| ds$$

$$= {}^*D_L(\hat{G}, F),$$

so ${}^*D_L(\hat{L}_F, {}^*f) \leq {}^*D_L(F, {}^*f) + 2D_L(\hat{G}, {}^*f) \approx 0$.

COROLLARY 31. *For any integrable* $f:[0,1]\to\mathbf{R}$, *the function* $L_f : T \to {}^*\mathbf{R}$ *defined by*

$$L_f(t) := |T| \cdot \int_t^{t+1/|T|} {}^*f(s)ds$$

is an S-integrable lifting of f. If f is continuous, L_f *is a uniform lifting of f, i.e. for all* $t \in T$, ${}^\circ L_f(t) = f({}^\circ t)$.

PROOF. L_f is a uniform lifting of f if f is continuous, since for continuous f for all $t \in T$ for all $s \in [t, t+1/|T|]$, ${}^*f(s) \approx f({}^\circ t)$.

COROLLARY 32. *For any measurable* $f:[0,1]\to\mathbf{R}$, *a lifting* $L_f' : T \to {}^*\mathbf{R}$ *is defined by*

$$L_f'(t) = {}^*e^{-1}\left(|T| \cdot \int_t^{t+1/|T|} {}^*e({}^*f(s))ds\right)$$

where $e:\mathbf{R}\to(-1,1)$, $e(x) = x(1+|x|)^{-1}$.

REMARK 33. The existence of liftings of measurable functions (see Theorem C (ii)), S-integrable liftings of integrable functions (see Theorem G (ii)), and uniform liftings of continuous functions has been shown by Anderson [1] and Keisler [4].

A direct proof of Corollary 31 (using standard measure theory instead of Theorem G) can be given by proving the claim for continuous f first and using the fact that the set of continuous functions is dense in $L^1([0,1])$.

We now apply Proposition 30 to stochastic processes. For any $f, g \in L_{\mathbf{R}}([0,1])$, $a \leqq b \in [0,1]$, if $f \sim g$ then

$$\int_a^b f(s)ds = \int_a^b g(s)ds,$$

so for any $\bar{f} \in \bar{L}_{\mathbf{R}}$, $a, b \in [0,1]$, it makes sense to define

$$I_{a,b}(\bar{f}) := \int_a^b f(s)ds.$$

Similarly, for $\bar{F} \in {}^*\bar{L}_{\mathbf{R}}$, $a < b \in {}^*[0,1]$, we write

$$I_{a,b}(\bar{F}) := \int_a^b F(s)ds.$$

We note that for any $f \sim g \in L_{\mathbf{R}}([0,1])$ the liftings L_f, L_g of f and g defined by Corollary 31 are identical.

COROLLARY 34. *Let* $x:\Omega\times[0,1]\to\mathbf{R}$ *be integrable,* $X^{\sim}:\Omega\to{}^*\bar{L}_{\mathbf{R}}([0,1])$ *a lifting of* $\tilde{x}:\Omega\to\bar{L}_{\mathbf{R}}([0,1])$. *Then the function* $X:\Omega\times T\to{}^*\mathbf{R}$ *defined by*

$$X(\omega,t):=|T|\cdot I_{t,t+1/|T|}(X^{\sim}(\omega))$$

is a lifting of x.

PROOF. X is clearly internal.

Since $X^{\sim}$ is a lifting of $\tilde{x}$, for almost all $\omega\in\Omega$, ${}^*\bar{D}_L(X^{\sim}(\omega),{}^*(\tilde{x}(\omega)))\approx 0$, so for almost all $\omega\in\Omega$, $X(\omega,\cdot)$ is a lifting of $x(\omega,\cdot)$ by Proposition 30. This implies that X is a lifting of x as shown in the proof of Theorem 24.

REMARK 35. For any integrable $x:\Omega\times[0,1]\to\mathbf{R}$, $\tilde{x}:\Omega\to\bar{L}_{\mathbf{R}}([0,1])$ has a lifting by Theorem C since $\tilde{x}$ is measurable (it is even integrable with respect to $\bar{D}_L$). This can be proved by standard methods or by proving the analogous versions of Theorem 24 and Corollary 25, for $L_M([0,1])$, which also gives a direct construction of a lifting of $\tilde{x}$ using an S-integrable lifting of x.

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